$SUSY-$APPROACH FOR INVESTIGATION OF TWO-DIMENSIONAL QUANTUM MECHANICAL SYSTEMS[a]

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Different ways to incorporate two-dimensional systems, which are not amenable to separation of variables, into the framework of Supersymmetrical Quantum Mechanics (SUSY QM) are analyzed. In particular, the direct generalization of one-dimensional Witten's SUSY QM is based on the supercharges of first order in momenta and allows to connect the eigenvalues and eigenfunctions of two scalar and one matrix Schrödinger operators. The use of second order supercharges leads to polynomial supersymmetry and relates a pair of scalar Hamiltonians, giving a set of such partner systems with almost coinciding spectra. This class of systems can be studied by means of new method of $SUSY-$separation of variables, where supercharges **allow** separation of variables, but Hamiltonians **do not**. The method of shape invariance is generalized to two-dimensional models to construct purely algebraically a chain of eigenstates and eigenvalues for generalized Morse potential models in two dimensions.

1. Introduction

Supersymmetric Quantum Mechanics [1], [2] is an interesting framework to analyze non-relativistic quantal problems. In particular, it allows to investigate the spectral properties of a wide class of quantum models as well as to generate new systems with given spectra. SUSY QM gives new insight into the problem of spectral equivalence of Hamiltonians, which historically was constructed as Factorization Method in Quantum Mechanics [3] and as Darboux-Crum transformations in Mathematical Physics [4].

During last two decades SUSY QM became an important and popular tool to study a wide variety of quantum systems (see the list of reports presented to this Conference). It is easy to note that the main stream of the development in SUSY QM concerned one-dimensional models. Though the variety of multi-dimensional (especially two- and three-dimensional) problems is much wider and practically important, much less attention has been given in the literature to

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the study of these models in SUSY QM. Thus the future progress seems to be mostly connected with investigation of its multi-dimensional generalizations. The main aim of this paper (based on the talk at the Conference) is to summarize different results of previous investigations of two-dimensional SUSY QM.

The paper is organized as follows. Two-dimensional generalization of the conventional Witten's formulation of SUSY QM is formulated in Section 2. In Section 3 the two-dimensional SUSY QM with the supercharges of second order in derivatives is presented. Section 4 contains a new, supersymmetric, approach for investigation of two-dimensional models, which **are not amenable** to separation of variables. This method is based on the second order supercharges introduced in the previous Section, and it gives a new opportunity to reduce the problem to the one-dimensional ones. Thus we obtain the specific method of $SUSY-$**separation of variables.** Section 5 is devoted to the generalization of the well known notion of shape invariance onto the two-dimensional models. Some new aspects typical for two-dimensional shape invariance are investigated. In Section 6 the deformation of SUSY QM algebra for models of Sections 3-5 is described.

2. Direct two-dimensional generalization of the conventional Witten's SUSY QM

The conventional one-dimensional SUSY QM was proposed by E.Witten [1]. It is characterized by the simplest realization of SUSY algebra:

$$\{\hat{Q}^+,\hat{Q}^-\} \;=\; \hat{H}; \quad (\hat{Q}^+)^2=(\hat{Q}^-)^2=0; \quad [\hat{H},\hat{Q}^{\pm}]=0, \tag{1}$$

where the superHamiltonian $\hat{H}$ is a diagonal matrix $\hat{H} = diag(h^{(0)},h^{(1)})$, with $h^{(i)} = -\partial^2 + V^{(i)}(x), \quad \partial \equiv d/dx$, and supercharges $\hat{Q}^{\pm}$ are off diagonal with elements $q^{\mp} \equiv \pm\partial+\partial W(x)$ of first order in derivatives with superpotential $W(x)$. In terms of components the (anti)commutation relations of SUSY algebra (1) mean, respectively, the factorization of Hamiltonians, nilpotent structure of supercharges and intertwining of $h^{(i)}$ by $q^{\pm}$. The superpotential $W(x)$ is defined by an arbitrary (possibly non-normalizable) solution $\Psi(x) \equiv \exp{(-W(x))}$ of the Schrödinger equation $h^{(0)}\Psi(x)=\epsilon\Psi(x)$ with $\epsilon \leq E_0^{(0)}$. If this solution is nodeless one has almost coinciding spectra of $h^{(0)}$ and $h^{(1)}$ or, equivalently, double degeneracy of the energy spectrum of H.

The direct multi-dimensional generalization of the construction above was built in [5] both by direct extension [6] of one-dimensional formulas and using [7] the superfield approach of Quantum Field Theory. We call it as "direct" since it retains both all relations of superalgebra (1) and the first order form of the components of supercharges.

Here we restrict ourselves to the particular case of **two space dimensions** $\vec{x} = (x_1, x_2)$. The precise formulas for the 4×4 superHamiltonian and supercharges are then the following:

$$\hat{H} = \begin{pmatrix} h^{(0)}(\vec{x}) & 0 & 0 \\ 0 & h^{(1)}_{ik}(\vec{x}) & 0 \\ 0 & 0 & h^{(2)}(\vec{x}) \end{pmatrix}; \quad i,k = 1,2; \quad \hat{Q}^+ = (\hat{Q}^-)^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q_1^- & 0 & 0 & 0 \\ q_2^- & 0 & 0 & 0 \\ 0 & p_1^+ & p_2^+ & 0 \end{pmatrix}; \qquad (2)$$

where two scalar Schrödinger operators $h^{(0)}, h^{(2)}$ and 2×2 matrix Schrödinger operator $h^{(1)}_{ik}$ are expressed in *quasifactorized* form in terms of the components of supercharges:

$$\begin{aligned} h^{(0)} &= q_l^+ q_l^- = -\partial_l^2 + V^{(0)}(\vec{x}) = -\partial_l^2 + (\partial_l W(\vec{x}))^2 - \partial_l^2 W(\vec{x}); \ \partial_l^2 \equiv \partial_1^2 + \partial_2^2; \\ h^{(2)} &= p_l^+ p_l^- = -\partial_l^2 + V^{(2)}(\vec{x}) = -\partial_l^2 + (\partial_l W(\vec{x}))^2 + \partial_l^2 W(\vec{x}); \\ h^{(1)}_{ik} &= q_i^- q_k^+ + p_i^- p_k^+ = -\delta_{ik}\partial_l^2 + \delta_{ik}((\partial_l W(\vec{x}))^2 - \partial_l^2 W(\vec{x})) + 2\partial_i\partial_k W(\vec{x}). \end{aligned}$$

The components of supercharges $q_l^\pm, p_l^\pm$ are again of first order and depend on the two-dimensional superpotential $W(\vec{x})$:

$$q_l^\pm \equiv \mp\partial_l + \partial_l W(\vec{x}); \quad p_l^\pm \equiv \epsilon_{lk} q_k^\mp. \qquad (3)$$

Two-dimensional 4×4 superHamiltonian and supercharges (2) realize the same conventional SUSY QM algebra (1). In components, the commutation relations in (1) are expressed as the intertwining relations between matrix $h^{(1)}_{ik}$ and $h^{(0)}, h^{(2)}$ by operators $q_l^\pm, p_l^\pm$:

$$h^{(0)} q_i^+ = q_k^+ h^{(1)}_{ki}; \quad h^{(1)}_{ik} q_k^- = q_i^- h^{(0)}; \quad h^{(1)}_{ik} p_k^- = p_i^- h^{(2)}; \quad p_k^+ h^{(1)}_{ki} = h^{(2)} p_i^+. \qquad (4)$$

The energy spectra of $h^{(0)}$ and $h^{(2)}$ are in general different, but the intertwining relations (4) provide the equivalence of energy spectra between a pair of two scalar Hamiltonians $h^{(0)}$, $h^{(2)}$ and 2×2 matrix Hamiltonian $h^{(1)}_{ik}$. The "equivalence" means coincidence of spectra up to zero modes of operators $q_l^\pm, p_l^\pm$. Thus the supersymmetry (supersymmetric transformation) allows to reduce the solution of matrix Schrödinger problem with the Hamiltonian $h^{(1)}_{ik}$ to solution of a couple of scalar Schrödinger problems $h^{(0)}$, $h^{(2)}$. Due to the same intertwining relations the vector wave functions of matrix Hamiltonian $h^{(1)}_{ik}$ are also connected (up to a normalization factor) with the scalar wave functions of scalar Hamiltonians $h^{(0)}, h^{(2)}$:

$$\begin{aligned} \Psi_i^{(1)}(\vec{x}; E) &= q_i^- \Psi^{(0)}(\vec{x}; E); \quad i = 1,2 \quad \Psi^{(0)}(\vec{x}; E) = q_i^+ \Psi_i^{(1)}(\vec{x}; E) \\ \Psi_i^{(1)}(\vec{x}; E) &= p_i^- \Psi^{(2)}(\vec{x}; E) \quad \Psi^{(2)}(\vec{x}; E) = p_i^+ \Psi_i^{(1)}(\vec{x}; E). \end{aligned}$$

The Schrödinger operators with matrix potential are not something very exotic in Quantum Mechanics. In particular, the described two-dimensional generalization of SUSY QM was successfully used [8] to investigate the spectra of Pauli operator for fermion in external electromagnetic fields. Nevertheless, considering this rather good-looking construction, one question seems

to be natural: is it possible to perform supersymmetric transformations in two-dimensional case avoiding any matrix Hamiltonians?

3. Second order supercharges in two-dimensional SUSY QM

The main idea, which could allow us to get rid of matrix components of superHamiltonian, is to explore the supercharges of second order in derivatives. For the first time such supercharges of higher orders in momenta were proposed for the one-dimensional situation in [9] (see also [10], [11]) leading to the polynomial deformation of SUSY algebra (see Section 6). In general, this approach implies a deformation of one relation of SUSY algebra (1) only, namely of the (quasi)factorization, but keeping unchanged the nilpotency of $\hat{Q}^{\pm}$ and the intertwining relations. The last seem to be the most important ingredient of SUSY methods in QM.

The simplest variant of second order supercharges $q^{\pm}$ - of the so called reducible [10] form - gives uninteresting result in two-dimensional case: the intertwined partner Hamiltonians differ by a trivial constant only, and both of them admit the separation of variables (see details in [12], [13]). By this reason here we will be interested in general irreducible second order components of supercharges:

$$q^{+} = g_{ik}(\vec{x})\partial_i\partial_k + C_i(\vec{x})\partial_i + B(\vec{x}); \quad q^{-} \equiv (q^{+})^{\dagger}. \tag{5}$$

The important question we have to investigate now concerns the existence of Hamiltonians

$$h^{(i)} = -\Delta^{(2)} + V^{(i)}(\vec{x}); \quad i = 1,2; \quad \Delta^{(2)} \equiv \partial_l\partial_l, \tag{6}$$

which satisfy the intertwining relations with $q^{\pm}$ of the form (5):

$$h^{(1)}q^{+} = q^{+}h^{(2)}; \quad q^{-}h^{(1)} = h^{(2)}q^{-}. \tag{7}$$

The first consequence of (7) restricts essentially the possible "metrics" $g_{ik}(\vec{x})$ by $\partial_l g_{ik} + \partial_i g_{lk} + \partial_k g_{il} = 0$ with solutions:

$$g_{11} = \alpha x_2^2 + a_1 x_2 + b_1;\; g_{22} = \alpha x_1^2 + a_2 x_1 + b_2;\; g_{12} = -\frac{1}{2}(2\alpha x_1 x_2 + a_1 x_1 + a_2 x_2) + b_3. \tag{8}$$

This has to be taken into account in rewriting [12] the intertwining relations (7) in components:

$$\partial_i C_k(\vec{x}) + \partial_k C_i(\vec{x}) + \Delta^{(2)} g_{ik}(\vec{x}) - (V^{(1)}(\vec{x}) - V^{(2)}(\vec{x}))g_{ik}(\vec{x}) = 0; \tag{9}$$

$$\Delta^{(2)} C_i(\vec{x}) + 2\partial_i B(\vec{x}) + 2g_{ik}(\vec{x})\partial_k V^{(2)}(\vec{x}) - (V^{(1)}(\vec{x}) - V^{(2)}(\vec{x}))C_i(\vec{x}) = 0; \tag{10}$$

$$\Delta^{(2)} B(\vec{x}) + g_{ik}(\vec{x})\partial_k\partial_i V^{(2)}(\vec{x}) + C_i(\vec{x})\partial_i V^{(2)}(\vec{x}) - (V^{(1)}(\vec{x}) - V^{(2)}(\vec{x}))B(\vec{x}) = 0. \tag{11}$$

The nonlinear system of second order differential equations (9) - (11) for unknown functions $C_i(\vec{x})$, $B(\vec{x})$, $V^{(1)}(\vec{x})$, $V^{(2)}(\vec{x})$ with constant parameters α, a_i, b_i in $g_{ik}(\vec{x})$ does not admit the general solution, but one can look for its particular solutions with concrete metrics g_{ik} and some ansatzes for unknown functions C_i.

In particular, the system (9) - (11) is essentially simplified for the metrics of elliptic form $g_{ik}(\vec{x}) \equiv \delta_{ik}$. In this case all unknown functions in (9) - (11) can be found analytically in the general form [12], but for all such solutions both Hamiltonians $h^{(1)}$, $h^{(2)}$ turn out to admit the $R-$separation [14] of variables in parabolic, elliptic or polar[c] coordinates, i.e. this class of two-dimensional problems can be reduced to two one-dimensional models.

Much more interesting situation appears for more complicated forms of metrics. Thus the list of particular solutions of the system (9) - (11) can be constructed analytically [12], [13], [15] for the hyperbolic metrics $g_{ik} = diag(+1, -1)$. Indeed, for this metrics a part of task of solution of (9) - (11) can be made in general form since it is reduced to a simpler system, which will be written in terms of coordinates $x_{\pm} \equiv x_1 \pm x_2 \quad \partial_{\pm} \equiv \partial/\partial x_{\pm}$ and $C_+ \equiv C_1 - C_2; \quad C_- \equiv C_1 + C_2$. The general solution can be provided by solving the system:

$$\partial_-(C_-F) = -\partial_+(C_+F); \tag{12}$$

$$\partial_+^2 F = \partial_-^2 F, \tag{13}$$

where $C_{\pm}$ depend only on $x_{\pm}$, respectively: $C_{\pm} \equiv C_{\pm}(x_{\pm})$. The function F, solution of (13), is represented as $F = F_1(x_+ + x_-) + F_2(x_+ - x_-)$. The potentials $V^{(1),(2)}(\vec{x})$ and the function $B(\vec{x})$ are expressed in terms of $F_1(2x_1)$, $F_2(2x_2)$ and $C_{\pm}(x_{\pm})$, solutions of (12):

$$\begin{aligned} V^{(1),(2)} &= \pm\frac{1}{2}(C_+' + C_-') + \frac{1}{8}(C_+^2 + C_-^2) + \frac{1}{4}\Big(F_2(x_+ - x_-) - F_1(x_+ + x_-)\Big); \qquad (14) \\ B &= \frac{1}{4}\Big(C_+C_- + F_1(x_+ + x_-) + F_2(x_+ - x_-)\Big), \end{aligned}$$

where C' means derivative in its argument.

For lack of a regular procedure for solution of both equations of the system (12), (13), its particular solutions can be found starting from certain anzatses for functions $C_{\pm}(x_{\pm}), F(\vec{x})$.

1) Let $C_- = 0$, then from (12) one obtains $F = \phi(x_-)/C_+(x_+)$. After inserting into Eq.(13) the separation of variables is possible, and particular solution reads[d]:

$$\begin{aligned} C_+(x_+) &= \frac{1}{\delta_1 \exp(\sqrt{\lambda}\cdot x_+) + \delta_2 \exp(-\sqrt{\lambda}\cdot x_+)}; \\ F_{1,2}(2x) &= \delta_1\sigma_{1,2}\exp(2\sqrt{\lambda}x) + \delta_2\sigma_{2,1}\exp(-2\sqrt{\lambda}x), \end{aligned}$$

the Greek letters – arbitrary constants, depending on sign of λ they may be real/complex.

[c]The reducible second order supercharges just correspond to separation of variables in polar coordinates.

[d]Here and below $F_{1,2}$ are defined only up to an arbitrary real constant: $F_1 \to F_1 + \epsilon,\ F_2 \to F_2 - \epsilon$.

2) Let $F(\vec{x})$ allows also the factorization: $F = F_+(x_+) \cdot F_-(x_-)$. Then from Eq.(12):

$$C_\pm = \frac{\nu_\pm}{F_\pm} \pm \frac{\gamma}{F_\pm} \int^{x_\pm} F_\pm dx'_\pm, \tag{15}$$

and there are two options to fulfill the condition (13), i.e. $F(\vec{x}) = F_1(2x_1) + F_2(2x_2)$:

$$a) \quad F_\pm(x_\pm) = \epsilon_\pm x_\pm, \qquad b) \quad F_\pm = \sigma_\pm \exp(\sqrt{\lambda} \cdot x_\pm) + \delta_\pm \exp(-\sqrt{\lambda} \cdot x_\pm). \tag{16}$$

Corresponding potentials can be found according to Eq.(14), being similar to ones obtained in [16] in quite different approach. Below some other solutions of (12), (13) will be built [15].

3) Let us start now from the general solution of (12):

$$F = L\Big(\int \frac{dx_+}{C_+} - \int \frac{dx_-}{C_-}\Big)/(C_+C_-). \tag{17}$$

Then Eq.(13) gives the functional-differential equation for the functional $L(A_+ - A_-)$ with $A'_\pm \equiv 1/C_\pm(x_\pm)$:

$$\Big(\frac{A'''_+}{A'_+} - \frac{A'''_-}{A'_-}\Big)L(A_+ - A_-) + 3(A''_+ + A''_-)L'(A_+ - A_-) + (A'^2_+ - A'^2_-)L''(A_+ - A_-) = 0, \tag{18}$$

where L' denotes the derivative of L with respect to its argument. If we take functions $A_\pm$ such that $A''_\pm = \lambda^2 A_\pm$, $\lambda = \mathrm{const}$, Eq.(18) will become ordinary differential equation for L with independent variable $(A_+ - A_-)$. It can be easily solved:

$$L(A_+ - A_-) = \alpha(A_+ - A_-)^{-2} + \beta,$$

where $A_\pm = \sigma_\pm exp(\lambda x_\pm) + \delta_\pm exp(-\lambda x_\pm)$ with $\sigma_+ \cdot \delta_+ = \sigma_- \cdot \delta_-$ and α, β - real constants. For $\lambda^2 > 0$, choosing $\sigma_\pm = -\delta_\pm = k/2$ or $\sigma_\pm = +\delta_\pm = k/2$, we obtain (up to an arbitrary shift in $x_\pm$) two particular solutions:

$$3a) \quad A_\pm = k\sinh(\lambda x_\pm), \quad 3b) \quad A_\pm = k\cosh(\lambda x_\pm).$$

Then (17) leads to:

$$3a) \qquad F_1(2x) = \frac{k_1}{\cosh^2(\lambda x)} + k_2 \cosh(2\lambda x); \tag{19}$$

$$F_2(2x) = \frac{k_1}{\sinh^2(\lambda x)} + k_2\cosh(2\lambda x); \quad C_\pm = \frac{k}{\cosh(\lambda x_\pm)}, \quad k \neq 0,$$

$$3b) \qquad F_1(2x) = -F_2(2x) = \frac{k_1}{\sinh^2(\lambda x)} + k_2\sinh^2(\lambda x), \quad C_\pm = \frac{k}{\sinh(\lambda x_\pm)}, \quad k \neq 0. \tag{20}$$

For $\lambda^2 < 0$ hyperbolic functions must be substituted by trigonometric ones.

We have to remark that the case $\lambda^2 = 0$, i.e. $A''_\pm = 0$, is not of interest, leading to trivial superpartners. However, choosing in (20) $\lambda \to 0, k,\ k_1,\ k_2^{-1} \to 0$ simultaneously, so that $\lambda^2 \sim k_1 \sim k_2^{-1} \sim k^2$, we obtain the solution:

$$F_1(2x) = -F_2(2x) = \tilde{k}_1 x^{-2} + \tilde{k}_2 x^2, \quad C_\pm = \frac{\tilde{k}}{x_\pm}. \tag{21}$$

One can check that (12) is also satisfied by

$$F_1(2x) = -F_2(2x) = k_1x^2 + k_2x^4, \quad C_\pm = \pm\frac{k}{x_\pm}. \tag{22}$$

4) Starting again from (17), it is convenient to pass on to new variable functions $C_\pm \equiv \pm f_\pm/f'_\pm$. Then F in (17) is represented in the form $F = U(f_+f_-)f'_+f'_-$ with an arbitrary[e] function U. After substitution in (13) one obtains the functional-differential equation:

$$(f'^2_+f^2_- - f^2_+f'^2_-)U''(f) + 3f\Big(\frac{f''_+}{f_+} - \frac{f''_-}{f_-}\Big)U'(f) + \Big(\frac{f'''_+}{f'_+} - \frac{f'''_-}{f'_-}\Big)U(f) = 0, \quad f \equiv f_+f_-.$$

For particular form of functions $f_\pm = \alpha_\pm exp(\lambda x_\pm) + \beta_\pm exp(-\lambda x_\pm)$, this equation becomes an ordinary differential equation for U with independent variable f. Its solution is $U = a + 4bf_+f_-$ ($a, b-$real constants). Then functions

$$\begin{aligned} F_1(x) &= k_1(\alpha_+\alpha_- \exp(\lambda x) + \beta_+\beta_- \exp(-\lambda x)) + k_2(\alpha_+^2\alpha_-^2 \exp(2\lambda x) + \beta_+^2\beta_-^2 \exp(-2\lambda x)), \\ -F_2(x) &= k_1(\alpha_+\beta_- \exp(\lambda x) + \beta_+\alpha_- \exp(-\lambda x)) + k_2(\alpha_+^2\beta_-^2 \exp(2\lambda x) + \beta_+^2\alpha_-^2 \exp(-2\lambda x)), \\ C_\pm &= \pm\frac{\alpha_\pm \exp(\lambda x_\pm) + \beta_\pm \exp(-\lambda x_\pm)}{\lambda(\alpha_\pm \exp(\lambda x_\pm) - \beta_\pm \exp(-\lambda x_\pm))} \end{aligned} \tag{23}$$

(with $k_1 \equiv a\lambda^2$, $k_2 \equiv 4b\lambda^2$) are real solutions of (12), (13), if $\alpha_\pm, \beta_\pm$ are real for $\lambda^2 > 0$, and $\alpha_\pm = \beta^*_\pm$ for $\lambda^2 < 0$.

5) To find a next class of solutions it is useful to rewrite (12) in terms of $x_{1,2}$:

$$(F_1(2x_1) + F_2(2x_2))\partial_1(C_+ + C_-) + F'_1(2x_1)(C_+ + C_-) + F'_2(2x_2)(C_+ - C_-) = 0.$$

Among known particular solutions the most compact one is:

$$C_+(x) = C_-(x) = ax^2 + c, \quad F_1(2x_1) = 0, \quad F_2(2x_2) = \frac{b^2}{x_2^2}. \tag{24}$$

After inserting these solutions (19) - (24) into the general formulas (14), one obtains the analytical expressions for potentials. Their explicit form can be found in [15].

The additional class of particular solutions of the system (9) - (11) obtained for the case of degenerate metrics $g_{ik} = diag(1, 0)$ can be found also in [15].

4. $SUSY-$ separation of variables

From the very beginning in this paper we are interested in two-dimensional quantum systems, which *are not amenable* to separation of variables. The supersymmetric approach, namely the intertwining relations (7), allows to formulate some specific supersymmetric alternative to the

[e] Due to Eq.(13), the function F should be additionally representable in the form $F = F_1(2x_1) + F_2(2x_2)$.

conventional notion of separation (including the so-called $R-$separation [14]) of variables. The main idea [17] is to consider such particular class of solutions of intertwining relations (7), when the components of supercharge $q^{\pm}$ *are amenable* to separation of variables but Hamiltonians $h^{(i)}$ *are not*. In this case the Hamiltonians $h^{(i)}$ turn out to be partially solvable, or in another terminology, quasi-exactly-solvable [18]. Both terms above mean that a part of spectrum (and possibly of corresponding eigenfunctions) of Hamiltonian is known. The crucial ingredient of the approach [17] is in the investigation of zero modes of intertwining operators $q^{\pm}$.

The general scheme of the method is the following. Let us suppose that $N+1$ normalizable zero modes of q^{+} are known (for example, due to separation of variables in q^{+}):

$$q^{+}\Omega_n(\vec{x}) = 0; \quad n = 0, 1, ..., N; \qquad q^{+}\vec{\Omega}(\vec{x}) = 0, \tag{25}$$

where $\vec{\Omega}(\vec{x})$ is a column vector with components $\Omega_n(\vec{x})$. From the intertwining relations (7) one can see that the space of zero modes is closed under the action of $h^{(2)}$, and therefore:

$$h^{(2)}\vec{\Omega}(\vec{x}) = \hat{C}\vec{\Omega}(\vec{x}), \tag{26}$$

where $\hat{C} \equiv ||c_{ik}||$ is a $c-$number $\vec{x}-$independent real matrix. If the matrix $\hat{C}$ can be diagonalized by a real similarity transformation:

$$\hat{B}\hat{C}(\hat{B})^{-1} = \hat{\Lambda} = diag(\lambda_0, \lambda_1, ..., \lambda_N), \tag{27}$$

the problem is reduced to a standard algebraic task within the zero modes space:

$$h^{(2)}(\hat{B}\vec{\Omega}(\vec{x})) = \hat{\Lambda}(\hat{B}\vec{\Omega}(\vec{x})). \tag{28}$$

It is not clear in advance, whether this general scheme is realized practically? To put it differently, are there any solutions $C_i(\vec{x}), B(\vec{x})$ of the intertwining relations (9) - (11), which give q^{+} with separation of variables?

To investigate this problem, it is useful [17] to transform the supercharge q^{+} by the special similarity transformation, which removes the terms linear in derivatives:

$$\tilde{q}^{+} = e^{(-\chi(\vec{x}))}q^{+}e^{(+\chi(\vec{x}))} = \partial_1^2 - \partial_2^2 + \frac{1}{4}(F_1(2x_1) + F_2(2x_2)); \ \chi(\vec{x}) = -\frac{1}{4}\Big(\int C_{+}(x_{+})dx_{+} + \int C_{-}(x_{-})dx_{-}\Big). \tag{29}$$

These new operators $\tilde{q}^{+}$ obviously obey the condition of separation of variables realizing the first step of our scheme of $SUSY-$separation of variables. Zero modes of $\tilde{q}^{+}$ can be found as linear superpositions of products of one dimensional wave functions $\eta_n(x_1)$ and $\rho_n(x_2)$, satisfying Schrödinger equations (with ϵ_n - the separation constants.):

$$(-\partial_1^2 - \frac{1}{4}F_1(2x_1))\eta_n(x_1) = \epsilon_n\eta_n(x_1); \qquad (-\partial_2^2 + \frac{1}{4}F_2(2x_2))\rho_n(x_2) = \epsilon_n\rho_n(x_2). \tag{30}$$

In analogy to (29), one can define operators

$$\tilde{h} \equiv \exp(-\chi(\vec{x}))\, h^{(2)}\, \exp(+\chi(\vec{x})) = -\partial_l^2 + C_1(\vec{x})\partial_1 - C_2(\vec{x})\partial_2 - \frac{1}{4}F_1(2x_1) + \frac{1}{4}F_2(2x_2), \quad (31)$$

and eigenfunctions of $\tilde{q}^+$ as:

$$\tilde{\Omega}_n(\vec{x}) = \exp(-\chi(\vec{x})) \cdot \Omega_n(\vec{x}), \quad (32)$$

keeping however in mind that the normalizability and orthogonality are not preserved automatically due to non-unitarity of the similarity transformation.

Then using (30) one can write:

$$\tilde{h}\tilde{\Omega}_n(\vec{x}) = [2\epsilon_n + C_1(\vec{x})\partial_1 - C_2(\vec{x})\partial_2]\tilde{\Omega}_n(\vec{x}). \quad (33)$$

It is not evident from (33), but the space spanned by functions $\tilde{\Omega}_n(\vec{x})$ is closed under the action of $\tilde{h}$. It will be demonstrated explicitly in the concrete model below.

In contrast to (29), where variables are separated, no separation for $\tilde{h}$, which would make the two-dimensional dynamics not-trivially reducible to one-dimensional dynamics. In this regard we refer to this method [17] for partial solvability as to $SUSY-$**separation of variables**.

The scheme of $SUSY-$ separation of variables formulated above can be used for arbitrary models satisfying the intertwining relations (7). The list of such models is already rather long, and it may increase in future, but it is very important to check the applicability of the scheme on the concrete model where the explicit solutions can be constructed. Actually, it means that solutions of two one-dimensional problems (30) can be found analytically. Below we briefly describe such a model - generalized two-dimensional Morse potential.

Among the solutions [15] of the system (12), (13) we focus attention on the particular case with a specific choice of parameters and $A > 0, \alpha > 0, a$–real constants [f]:

$$C_+ = 4a\alpha; \quad C_- = 4a\alpha \cdot \coth\frac{\alpha x_-}{2}; \quad (34)$$

$$f_i(x_i) \equiv \frac{1}{4}F_i(2x_i) = -A\Big(e^{-2\alpha x_i} - 2e^{-\alpha x_i}\Big); \quad i = 1,2; \quad (35)$$

$$V^{(1),(2)} = \alpha^2 a(2a \mp 1)\sinh^{-2}\Big(\frac{\alpha x_-}{2}\Big) + 4a^2\alpha^2 + A\Big[e^{-2\alpha x_1} - 2e^{-\alpha x_1} + e^{-2\alpha x_2} - 2e^{-\alpha x_2}\Big] \quad (36)$$

One easily recognizes in (36) a sum of two Morse potentials plus a hyperbolic singular term which prevents to apply the *conventional* methods of separation of variables. These singular terms can be both attractive, for $|a| > \frac{1}{2}$, or one repulsive and one attractive, for $|a| < \frac{1}{2}$. The parameter a will be further constrained by the condition that the strength of the attractive singularity at $x_- \to 0$ should not exceed the well known bound $-1/(4x_-^2)$.

[f] For the complexification of the model see [19]

The normalizable functions $\tilde{\Omega}_n(\vec{x})$ (and $\Omega_n(\vec{x})$) can be constructed from the well known [20] normalizable solutions of (30) with $\epsilon_n < 0$:

$$\tilde{\Omega}_n(\vec{x}) = \exp(-\frac{\xi_1+\xi_2}{2})(\xi_1\xi_2)^{s_n}F(-n,2s_n+1;\xi_1)F(-n,2s_n+1;\xi_2), \tag{37}$$

where $F(-n,2s_n+1;\xi)$ is the standard degenerate (confluent) hypergeometric function, reducing to a polynomial for integer n, and

$$\xi_i \equiv \frac{2\sqrt{A}}{\alpha}\exp(-\alpha x_i); \quad s_n = \frac{\sqrt{A}}{\alpha} - n - \frac{1}{2} > 0; \quad \epsilon_n = -A\Big[1 - \frac{\alpha}{\sqrt{A}}(n+\frac{1}{2})\Big]^2. \tag{38}$$

The number $(N+1)$ of normalizable zero modes (37) is determined by the inequality $s_n > 0$.

The condition of normalizability of zero modes $\tilde{\Omega}_n(\vec{x})$, together with the absence of the "fall to the centre", leads [17] to the following two ranges of parameters:

$$a \in (-\infty, -\frac{1}{4} - \frac{1}{4\sqrt{2}}); \quad s_n = \frac{\sqrt{A}}{\alpha} - n - \frac{1}{2} > -2a > 0. \tag{39}$$

$$a \in (-\frac{1}{4}, \frac{1}{4}); \quad s_0 > 2(|a|+1). \tag{40}$$

In Sections 4 and 5 only the region (39) will be considered. Inequalities (39) can be satisfied by the choice of a and A, and/or by suitable restriction on the number N of zero modes $\Omega_n(\vec{x})$.

Analysis of the action of $\tilde{h}$ in (33) gives that the matrix $\hat{C}$ is of triagonal form. It can be diagonalized explicitly by a similarity transformation, and the eigenvalues E_k of $h^{(2)}$ coincide with its (all different and nonzero) diagonal elements:

$$E_k = c_{kk} = -2(2a\alpha^2 s_k - \epsilon_k). \tag{41}$$

The resulting eigenfunctions of $h^{(2)}$ are obtained (see Eqs.(27), (28)) from the constructed zero modes $\tilde{\Omega}_n(\vec{x})$ and the similarity transformation $\hat{B}$:

$$\Psi_{N-n}(\vec{x}) = \Sigma_{l=0}^{N} b_{nl}\Omega_l(\vec{x}). \tag{42}$$

For the algorithm of iterative construction of coefficients b_{nl} see [17]. Thus the construction of the set of eigenfunctions, which lie in the space of zero modes, is completed.

These eigenfunctions $\Psi_k(\vec{x})$ may be also used for constructing more general eigenfunctions of $h^{(2)}$ via a product ansatz:

$$\Phi(\vec{x}) \equiv \Psi_k(\vec{x}) \cdot \Theta(\vec{x}). \tag{43}$$

Three such eigenfunctions based on Ψ_0 were constructed in [17]. Within the bounds imposed (39) only one of them is normalizable, though for the region (40) all three are normalizable.

5. Shape invariance in two dimensions

In the previous Section we developed the method which led to construction of the partially solvable (quasi-exactly-solvable) two-dimensional models. Let us remind now the well known *in one dimension* and very elegant method of **shape invariance** [21] usually associated with the exactly solvable one-dimensional systems. Our aim here is to generalize the idea of shape invariance to the two-dimensional case [17].

Refering readers to the original paper [21] and reviews [2] for the detail discussion of standard one-dimensional shape invariance method, let us list its main steps only:

$$\tilde{H}(x;a) = H(x;\bar{a}) + \mathcal{R}(a); \quad \bar{a} = \bar{a}(a) \tag{44}$$

where $\mathcal{R}(a)$ is a (c-number) function of a. The absence of spontaneous breaking of supersymmetry for all values of a implies that the lowest eigenvalue $E_0(a)$ of $H(a)$ vanishes and the corresponding eigenfunctions $\Psi_0(a)$ are normalizable zero modes of $Q^+(a)$.

The intertwining relations $Q^-(x;a)\tilde{H}(x;a) = H(x;a)Q^-(x;a)$ with the standard first order supercharge allow in this case to solve the entire spectral problem for $H(x;a)$. The crucial steps are as follows.

$$H(x;\bar{a})\Psi_0(x;\bar{a}) = E_0(\bar{a})\Psi_0(x;\bar{a}) = 0; \quad \tilde{H}(x;a)\Psi_0(x;\bar{a}) = \mathcal{R}(a)\Psi_0(x;\bar{a}). \tag{45}$$

It is important to remark that $\Psi_0(x;\bar{a}) \equiv \tilde{\Psi}_0(x;a)$ has no nodes and therefore is the ground state wave function of $\tilde{H}(x;a)$. Then

$$H(x;a)\Big[Q^-(x;a)\Psi_0(x;\bar{a})\Big] = \mathcal{R}(a)\Big[Q^-(x;a)\Psi_0(x;\bar{a})\Big]. \tag{46}$$

Provided $\Big[Q^-(xa)\Psi_0(x;\bar{a})\Big]$ is normalizable, we have generated an excited state of $H(x;a)$, and thus $\mathcal{R}(a)$ is naturally positive. It is clear that these steps can be repeated up to the last one, where the resulting wave function Ψ will no more be normalizable.

It is also clear that the isospectrality of $H(x;a)$ and $\tilde{H}(x;a)$ (up to the only zero mode $\Psi_0(x;a)$) implies that there is no eigenvalue of $H(x;a)$ between zero and the ground state energy $\tilde{E}_0(a)$ of $\tilde{H}$. This observation leads to a proof that after suitable iterations one gets the entire spectrum of $H(x;a)$. This method is referred as algebraic solvability (or complete solvability) by shape invariance in one-dimensional SUSY QM.

To proceed to the formulation of two-dimensional shape invariance, we start from the relatively simple two-dimensional case of systems with conventional separation of variables:

$$H(\vec{x}) = H_1(x_1) + H_2(x_2); \quad \vec{x} = (x_1, x_2).$$

Now suppose that H_1 and H_2 both are shape invariant:

$$\tilde{H}_i(x_i;a_i) = H_i(x_i;\bar{a}_i) + \mathcal{R}_i(a_i) \quad \leftrightarrow \quad \tilde{H}(\vec{x};\mathbf{a}) = H(\vec{x};\bar{\mathbf{a}}) + \mathcal{R}(\mathbf{a});\ \mathbf{a} \equiv (a_1, a_2). \tag{47}$$

In order to realize a nontrivial intertwining relations for $H, \tilde{H}$ one can consider factorized supercharges of second order written as products of first order supercharges:

$$Q^{\pm} = Q_1^{\pm} \cdot Q_2^{\pm}; \quad Q_i^{\pm} = \mp\partial_i + W_i(x_i). \tag{48}$$

There is the considerable difference with respect to the one-dimensional case. The crucial reason is that the space of zero modes of supercharges becomes now of higher dimensionality including the products of one-dimensional zero modes of the first Hamiltonian times all states of the second Hamiltonian and vice versa. While iterations are again obviously possible, it is clear that one can not argue about the entire solvability of the spectral problem, because in general many zero modes of (48) exist. Their number depends on the confining properties of H_1 and H_2. For example, in a case of oscillator-like potentials this number becomes infinite, and they are distributed over the whole spectrum. In this case only partial solvability of H can be achieved by the choice of (48) and shape invariance. Of course, one can solve such trivial models by separate use of $Q^{\pm} = Q_i^{\pm}$, which allows to solve the entire spectrum of two-dimensional model in terms of the one-dimensional ones.

Let us suppose to have two-dimensional system (without separation of variables) with a Hamiltonian H, which is related to $\tilde{H}$ by (44). For simplicity (in general, there is no connection between the dimensionality of the Schrödinger equation and the dimensionality of the parameter manifold), we assume that shape invariance is realized with one parameter a. Two-dimensional SUSY QM does not identify here zero modes of $Q^{\pm}$ with the ground state of the Hamiltonian. Thus one has to repeat the steps (44) - (46) above by taking into account $E_0(a) \neq 0$. In order to make our discussion more explicit we will from now on refer explicitly to the model (34) - (36) ($H \equiv h^{(2)}$), $\tilde{H} \equiv h^{(1)}$ with the parameter a being bound to (39).

First of all we observe that this model is indeed shape invariant (the infinite domain given by (39) allows iterations of (44)):

$$\bar{a} = a - \frac{1}{2}; \quad \mathcal{R}(a) = \alpha^2(4a - 1)) \tag{49}$$

The starting point is to write (46):

$$H(\vec{x}; a)\Big[Q^-(\vec{x}; a)\Psi_0(\vec{x}; a - \frac{1}{2})\Big] = \Big(E_0(a - \frac{1}{2}) + \mathcal{R}(a)\Big) \cdot \Big[Q^-(\vec{x}; a)\Psi_0(\vec{x}; a - \frac{1}{2})\Big], \tag{50}$$

where $E_0(a)$ and $\Psi_0(\vec{x}; a)$ not to be identified with ground state. Thus we have constructed the new eigenstate and eigenvalue of $H(\vec{x}; a)$, provided $Q^-(\vec{x}; a)\Psi_0(\vec{x}; a - \frac{1}{2})$ is normalizable. Note that the eigenvalue $\Big(E_0(a - \frac{1}{2}) + \mathcal{R}(a)\Big)$ is larger than $E_0(a)$ with the bounds of (39).

It is interesting that the energy of the first iteration of shape invariance in (50) coincides precisely with the eigenvalue of the additional solution $\Phi(\vec{x})$ mentioned in the previous Section:

$$E = E_0(a - \frac{1}{2}) + \mathcal{R}(a) = \alpha^2[4a(1 - s_0) + (2s_0 - 1)] + 2\epsilon_0.$$

The next iteration of shape invariance will give:

$$H(a)\Big[Q^-(a)Q^-(a-\frac{1}{2})\Psi_0(a-1)\Big] = \Big(E_0(a-1)+\mathcal{R}(a-\frac{1}{2})+\mathcal{R}(a)\Big)\cdot\Big[Q^-(a)Q^-(a-\frac{1}{2})\Psi_0(a-1)\Big], \tag{51}$$

and the new eigenfunction $Q^-(a)Q^-(a-\frac{1}{2})\Psi_0(a-1)$ can be written explicitly as function of $\vec{x}$. Provided normalizability is ensured, one can thereby construct a chain by successive iterations of (50) and (51), since $Q^-(a)$ has no normalizable zero modes in (39). The end point of such a chain will be given by non-normalizability of the relevant wave function.

Let us stress that though we illustrated both methods by the same model, $SUSY-$separation of variables can be implemented completely independently from shape invariance. For example, the model considered in the range (40) admits the method of $SUSY-$separation of variables, but shape invariance can not be applied since the domain (40) is too small.

6. Polynomial algebra for two-dimensional SUSY QM

In this last Section we will analyze the deformation of conventional SUSY QM algebra (1) due to introducing of second order components $q^\pm$ of supercharges in Sections 3-5. It is obvious that keeping the intertwining relations (7) and the matrix structure of supercharges $\hat{Q}^\pm$ we do not change two relations of SUSY algebra (1), which express the supersymmetry of $\hat{H}$ and nilpotency of $\hat{Q}^\pm$. But the third relation, which gives the (quasi)factorization of the components of $\hat{H}$, cannot be fulfilled. In the one-dimensional case the anticommutator of $\hat{Q}^\pm$ gives [10] the operator of fourth order in derivatives which can be represented as a second order polynomial of the superHamiltonian $\hat{H}$. The situation changes crucially for two-dimensional systems of Sections 3-6, where in general a new *diagonal* operator of fourth order appears:

$$\hat{R} \equiv \{\hat{Q}^+, \hat{Q}^-\} \tag{52}$$

This operator obviously commutes with the superHamiltonian $\hat{H}$ due to supersymmetry of $\hat{H}$ (intertwining relations (7)). It is shown in [13] that for Laplacian metrics in (5) $g_{ik} = \delta_{ik}$ (where variables can be separated) operator $\hat{R}$ can be reduced to the second order symmetry operator $\tilde{R}$ up to a second order polynomial (with constant coefficients) of $\hat{H}$. But for all other metrics g_{ik}, including hyperbolic and degenerate ones, the Theorem was proved [13]: *The symmetry operator $\hat{R}$ is essentially of fourth order in derivatives, its order can not be reduced.* The components $R^{(i)}$, $i = 1, 2$ of $\hat{R}$ are the symmetry operators of the components $h^{(i)}$: $[h^{(i)}, R^{(i)}] = 0$. The explicit expressions of $R^{(i)}$ for known solutions of (9) - (11) can be written (see [22]). Thus all two-dimensional models of Section 3 are **integrable**[g].

[g]We note that integrability of the system does not mean in general its (even partial) solvability.

As for generalized Morse model of Section 4, the quantum integral of motion $R = Q^-Q^+$ gives zero on the eigenfunctions $\Psi_k(\vec{x})$ by construction, since they are zero modes of Q^+. But by a direct calculation one can check that all three additional eigenfunctions $\Phi(\vec{x})$ (see (43)) of H are simultaneously [17] eigenfunctions (with nonzero eigenvalues) of the symmetry operator R. Thus they belong to a system of common eigenfunctions of two Hermitian mutually commuting operators H and R.

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